ABSTRACT

HUGGINS, J. FRANTZ B.M.E., U.D., Dayton, Ohio, 1981
B.S., W.U., Wilberforce, Ohio, 1981

APPLICATION OF HARMONIC BALANCE TO A TRULY NONLINEAR OSCILLATOR EQUATION

Advisor: Professor Ronald E. Mickens

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We investigate the solutions of a "truly" nonlinear, one-dimensional oscillator equation. From general considerations, we expect the system to have a stable limit cycle. An estimation of the parameters which characterize the limit cycle is obtained using the method of harmonic balance. A detailed discussion of this approximate analytical solution is given along with possible future investigations.

APPLICATION OF HARMONIC BALANCE TO A
TRULY NONLINEAR OSCILLATOR EQUATION

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J. FRANTZ HUGGINS

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DEDICATION

This work is dedicated to my Lord and Savior Jesus Christ who gave me the strength and courage to overcome my obstacles.

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CHAPTER ONE

INTRODUCTION

Nonlinear differential equations arise in all areas of science, technology and engineering.[1]

This thesis is a continuation of the research program started by Professor Ronald Mickens to investigate the use of the method of harmonic balance for obtaining approximate analytical solutions to second-order nonlinear differential equations.[2,3,4,5] The details of this method are given in Mickens[2] and summarized in the Appendix.

Harmonic balance has been applied to the following equations

$$\ddot{x} + x^3 = 0, \tag{1}$$

$$\ddot{x} + x = \epsilon \left[1 - \frac{\beta}{1-x^2}\right]\dot{x}, \tag{2}$$

$$\ddot{x} + x = \epsilon f(x,\dot{x}), \tag{3}$$

where ϵ is a small positive parameter, β is a constant, and $f(x,\dot{x})$ is a polynomial function of its arguments. The system described by eq. (1) is conservative and has the potential energy function

$$V(x) = \frac{x^4}{4}. \tag{4}$$

Inspection shows that this potential has only bound states, consequently, all the motions are periodic. However, the frequency depends on the amplitude. In fact, as Hill[3] showed, the frequency is given by the relation

$$\omega(A) = \left(\frac{3}{4}\right)^{1/2} A. \tag{5}$$

This result is correct to within about 5%.

Equation (2) is a model for a new type of nonlinear electronic oscillator circuit.[6] Note that it is singular when $x = \pm 1$. In general, we do not expect the usual techniques[1], such as perturbation theory, method of slowly varying amplitude and phase, and multi-time procedures, to work in this situation. However, Bota and Mickens[5] showed that harmonic balance gave excellent results for values of the initial amplitudes as close as within 5% of the singular points $x = \pm 1$. The parameters of the limit cycle were in agreement with those obtained from an accurate numerical integration of eq. (2).

For eq. (3), if f is a function of both x and $\dot{x}$, then we should expect the existence of limit cycles and a limit point. Again, the application of the method of harmonic balance gives results in agreement with those obtained by other techniques.[1,2]

It should be pointed out that harmonic balance is not limited to equations with small non-linear terms. In many cases, it provides good results for large non-linear terms. In fact, it can give the exact solution under certain conditions, namely, when the differential equation under consideration has an exact steady-state solution of the form $x = A \cos \omega t$.

To date, the major conclusion is that the method of harmonic balance is a fast and efficient technique for investigating the oscillatory solutions of non-linear differential equations provided certain applicability conditions are satisfied. (See Mickens[2] and the Appendix.)

Before we turn to the particular nonlinear equation considered in this thesis, the following related information should be given. In a recent paper, Mickens and Oyedeji[4] investigated the construction of approximate analytical solutions to a new class of nonlinear oscillator equation

$$\ddot{x} + x^3 = \epsilon f(\dot{x}, x), \qquad (6)$$

where ϵ is a positive, small parameter and f is a polynomial function of its arguments. Note that this is a truly nonlinear oscillator since for $\epsilon = 0$, we get the equation looked at by Hill.[3] What Mickens and Oyedeji showed was that an approximate solution having the form

$$x(t) = A(t)\cos\left[\omega t + \phi(t)\right] , \tag{7}$$

could be found where

$$\dot{A}(t) = -\left(\frac{\epsilon}{2\pi\omega}\right)\int_0^{2\pi} f \sin\psi d\psi \tag{8}$$

$$A\dot{\phi}(t) = -\left(\frac{\epsilon}{2\pi\omega}\right)\int_0^{2\pi} f \cos\psi d\psi + \frac{A}{2}\left[\frac{3A}{4\omega} - \omega\right]. \tag{9}$$

In equations (8) and (9), the function f under the integral sign is

$$f = f(A\cos\Psi , -\omega A \sin\Psi). \tag{10}$$

These results are a generalization of the method of slowly varying amplitude and phase for the equation

$$\ddot{x} + x = \epsilon f(x,\dot{x}), \tag{11}$$

to the eq. (6). We would not expect this technique to hold for f $(x,\dot{x})$ being a rational function of x and $\dot{x}$.

In this thesis, we apply the method of harmonic balance to the equation

$$\ddot{x} + x^3 = \epsilon \left[1 - \frac{\beta}{1 - x^2} \right] \dot{x}, \tag{12}$$

where $0 < \epsilon << 1$ and β is a constant. We are interested in this equation for several reasons:

(i) It is an example of a singular, nonlinear oscillatory differential equation.[5] Therefore, the usual techniques are not expected to yield approximate analytical solutions.

(ii) This equation is a truly nonlinear oscillator, since $\epsilon = 0$, gives Hill's equation.

(iii) The above two properties make it an interesting proposition as to whether harmonic balance will work for this case.

There does not exist a theory of the method of harmonic balance beyond the first-order results. In general, one can show that harmonic balance in the first approximation gives the same result as the method of perturbation. No consistent, higher-order corrections are presently known.

In Chapter Two, we calculate an approximate analytical solution to equation (12) using the method of harmonic balance. Chapter Three discusses this solution and derives the limitations in parameter space of this solution. Finally, in Chapter Four, we present several ideas for future investigations.

CHAPTER TWO

AN APPROXIMATE ANALYTICAL SOLUTION

According to the method of harmonic balance, the equation

$$\ddot{x} + x^3 = \epsilon\left[1 - \frac{\beta}{1 - X^2}\right]\dot{x}, \tag{13}$$

is assumed to take the form

$$x = A\cos\omega t, \quad x(0) = A, \quad x(0) = 0, \tag{14}$$

where A and ω are a priori unknown. Since the right-side of eq. (13) depends on both x and $\dot{x}$, we expect limit cycle behavior. Corresponding to the assumed solution, given by eq. (14), we have the higher derivatives and nonlinear term

$$\dot{x} = -\omega A \sin\omega t, \tag{15a}$$

$$\ddot{x} = -\omega^2 A \cos\omega t, \tag{15b}$$

$$x^3 = \frac{A^3}{4}(3\cos\omega t + \cos 3\omega t). \tag{15c}$$

Substitution of eqs. (14), (15a), (15b) and (15c) into eq.

(13), gives, on simplification

$$\epsilon \omega A \left[1 - \beta - \frac{A^2}{4} \right] \sin \omega t$$
$$- A \left[\omega^2 - \frac{3A^2}{4} (1 + \omega^2) - 5/8\ A^4 \right] \cos \omega t$$
$$+ \text{(higher-order harmonic)} = 0 . \qquad (16)$$

Setting the coefficients for the lowest harmonics equal to zero gives

$$1 - \beta - \frac{A^2}{4} = 0. \qquad (17)$$

$$\omega^2 - \frac{3A^2}{4} (1 + \omega^2) - \frac{5}{8} A^4 = 0. \qquad (18)$$

Therefore, from eq. (17), we obtain

$$A = 2 \left(1 - \beta \right)^{\frac{1}{2}} \qquad (19)$$

Solving eq. (18) for ω^2 gives

$$\omega^2 = \frac{\dfrac{5A^4}{8} - \dfrac{3A^2}{4}}{\dfrac{3A^2}{4} - 1}$$

$$= \frac{5A^4 - 6A^2}{6A^2 - 8}$$

$$= \frac{A^2 \, (5A^2 - 6)}{6A^2 - 8} \, . \qquad (20)$$

Now substituting eq. (19) into eq. (20) and simplifying the resulting expression give

$$\omega^2 = \frac{(1 - \beta)\,(7 - 10\beta)}{(2 - 3\beta)} \, . \qquad (21)$$

Note that A and ω depend only on β; there is no dependence on ϵ. Also, the A and ω obtained are the amplitude and frequency of the (steady-state) limit cycle.

The approximate analytical solution to eq. (13) according to the method of harmonic balance is obtained by the substitution of eqs. (19) and (21) into the assumed solution of eq. (14). Doing this, gives

$$x = 2\left(1 - \beta\right)^{\frac{1}{2}} \cos \left[\frac{(1 - \beta)\quad(7 - 10\beta)}{(2 - 3\beta)}\right]^{\frac{1}{2}} t \qquad (22)$$

We wish to indicate that the related singular, nonlinear equation

$$\ddot{x} + x = \epsilon \left[1 - \frac{\beta}{1 - X^2} \right] , \qquad (23)$$

is, in fact, the large amplitude generalization of the Van der Pol equation. To show this, note that

$$1 - \frac{\beta}{1 - X^2} = 1 - \beta(1 + x^2) + 0(x^3)$$

$$= (1 - \beta) - \beta x^2 + 0(x^3). \qquad (24)$$

Neglecting terms of order x^3 and higher, gives, on substitution of eq. (24) into (23)

$$\ddot{x} + x = \epsilon \left[(1 - \beta) - \beta x^2 \right] \dot{x}, \qquad (25)$$

which on rescaling of the variables is just the Van der Pol equation.

Therefore, we can conclude that eq. (13) is the large amplitude generalization of the following equation

$$\ddot{x} + x^3 = \epsilon (1 - x^2) \dot{x}, \qquad (26)$$

with the variables rescaled.

CHAPTER THREE

DISCUSSION

We now discuss the implications of equations (19) and (21). This will lead us to restrictions on the parameter ranges for which the method of harmonic balance works.

Consideration of eq. (13) shows that the equation is not defined for $x = \pm 1$. Physically, these singular points correspond to the system having an infinite amount of energy, i.e., the system can have arbitrarily large displacements and velocities. For this reason, we place the following restriction on the amplitude

$$|A| < 1 \qquad (27)$$

Since the ranges of A values

$$0 < A < 1, \qquad (28)$$

and

$$-1 < A < 0 \qquad (29)$$

are equivalent, up to a phase, we write our restriction as that given by eq. (28). Therefore, from eq. (19)

$$0 < 2 \left[1 - \beta\right]^{\frac{1}{2}} < 1, \tag{30}$$

we obtain from the lower limit

$$2 \left[1 - \beta\right]^{\frac{1}{2}} > 0, \tag{31}$$

the bound

$$\beta < 1, \tag{32}$$

and from the upper limit

$$2 \left[1 - \beta\right]^{\frac{1}{2}} < 1$$

$$4 \quad (1 - \beta) < 1$$

$$1 - \beta < 1/4$$

the bound

$$\beta > 3/4. \tag{33}$$

The frequency squared, ω^2 , has to be positive; therefore,

$$\omega^2 = \frac{(1 - \beta)\ (7 - 10\beta)}{(2 - 3\beta)} > 0\ . \tag{34}$$

There are several cases to consider. First if $(2 - 3\beta) > 0$, then

$$(1 - \beta)\ (7 - 10\beta) > 0. \qquad (35)$$

From eq. (32), we conclude that

$$7 - 10\beta > 0 \qquad (36)$$

or

$$\beta < 7/10. \qquad (37)$$

However, this latter result contradicts the inequality of eq. (33).

Second, let $(2 - 3\beta) < 0$, then

$$7 - 10\beta < 0 \qquad (38)$$

and

$$\beta > 7/10. \qquad (39)$$

Combining, the inequalities of eqs. (32), (33) and (39), we finally conclude that

$$3/4 < \beta < 1, \qquad (40)$$

if the method of harmonic balance is to apply to the nonlinear, singular differential expression of eq. (13).

Numerical integration of both eqs. (13) and (23) indicate that solutions exist, for $0 < \epsilon \ll 1$, only for initial amplitudes less than one in absolute value and for β in the range given by eq. (40). In more detail, we have

$$A = 2\left[1 - \beta\right]^{\frac{1}{2}} + 0(\epsilon), \tag{41}$$

$$\omega^2 = \frac{(1-\beta)\ (7 - 10\beta)}{(2 - 3\beta)} + 0(\epsilon), \tag{42}$$

that is both A and ω depend on ϵ and β. However, in the first approximation, neither one depends on ϵ. It is only in higher-order approximations that the ϵ dependency will be seen. Again, numerical integration of eqs. (13) and (23) clearly show this to be true. (See Bota and Mickens.[5])

Finally, it should be pointed out that eq. (23) has the following solution obtained from the method of harmonic balance

$$x\ (t) = 2\left[1 - \beta\right]^{\frac{1}{2}} \cos t. \tag{43}$$

This corresponds to

$$A = 2\left[1 - \beta\right]^{\frac{1}{2}} + 0(\epsilon), \tag{44}$$

and

$$\omega = 1 + 0(\epsilon). \tag{45}$$

CHAPTER FOUR

FUTURE INVESTIGATIONS

The following areas would prove fruitful for possible future investigations on the method of harmonic balance as well as the particular nonlinear differential equations indicated:

(i) Consider the general class of nonlinear differentiation equations

$$\ddot{x} + x^{2N + 1} = \epsilon f\ (x,\dot{x}), \tag{46}$$

where $0 < \epsilon << 1$, N is a positive integer and f can be either a "regular" or singular function of its arguments. Based on the cases for N = 0 and N = 1, it would seem that harmonic balance should provide a good approximate analytical solution. Question: Can one look at this entire class of equations and investigate the class rather than the individual members of the class?

(ii) If $f(x,\dot{x})$ is the singular function

$$f(x,\dot{x}) = 1 - \frac{\beta}{1 - X^2} \tag{47}$$

then, does the limit-cycle amplitude have the same value, for eq. (46), as that obtained for N = 0 and N = 1? That is, does

$$A = 2 \left[1 - \beta\right]^{\frac{1}{2}}, \tag{48}$$

for all positive integers, in the case of eq. (46)?

(iii) Can uniformly valid, second-order corrections be made to the method of harmonic balance.

(iv) Finally, can the method of harmonic balance be generalized so that the transient behavior of the nonlinear system can be determined?

APPENDIX

METHOD OF HARMONIC BALANCE

The method of harmonic balance applies to the determining of solutions to nonlinear differential equations which have periodic solutions that are close to harmonic. Details of the method are given in Mickens[2]. Here, we merely outline the method.

The solution

$$x(t) = A \cos \omega t, \tag{A.1}$$

where A and ω are *a priori* unknown are substituted into the differential equation

$$F(\ddot{x}, \dot{x}, t, \alpha) = 0, \tag{A.2}$$

where α represents the collection of parameters which define the differential equation. F is a nonlinear function of its arguments. After some algebraic manipulations, the following expression is obtained

$$g(A, \omega, \alpha) \cos \omega t + h(A, \omega, \alpha) \sin \omega t + (\text{higher-order harmonics}) = 0 \tag{A.3}$$

The coefficients of the lowest harmonic are then set equal to zero

$$g\ (A, \omega, \alpha) = 0\ , \quad h\ (A, \omega, \alpha) = 0. \qquad (A.4)$$

These two equations can be solved to give

$$A = \dot{A}\ (\alpha)\ , \quad \omega = \omega(\alpha). \qquad (A.5)$$

Multi-solutions give rise to a system where there are several limit cycles/limit point. Finally, the solution, according to the method of harmonic balance is,

$$X\ (t, \alpha) \simeq A\ (\alpha)\ \cos \omega(\alpha)\ t. \qquad (A.6)$$

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